*Journal of Statistical Physics, Vol. 56, Nos. 1/2, 1989*

# On the Critical Exponent for Random Walk Intersections

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*Received November 16, 1988; final January 30, 1989*

The exponent $\zeta_d$ for the probability of nonintersection of two random walks starting at the same point is considered. It is proved that $1/2 < \zeta_2 \leqslant 3/4$. Monte Carlo simulations are done to suggest $\zeta_2 = 0.61...$ and $\zeta_3 \approx 0.29$.

**KEY WORDS:** Random walks; intersections; critical exponents.

## 1. INTRODUCTION

Let $S^1$, $S^2$ be independent simple random walks starting at the origin in $\mathbb{Z}^d$ and let $f(n)$ be the probability that the paths of the first $n$ steps do not intersect, i.e.,

$$f(n) = P\{S^1[0, n] \cap S^2(0, n] = \varnothing\}$$

where $S^j[a, b] = \{S^j(i): a \leqslant i \leqslant b\}$, $S^j(a, b] = \{S^j(i): a < i \leqslant b\}$. For $d \leqslant 4$, $f(n) \to 0$ as $n \to \infty$,[4] while for $d > 4$, $f(n) \to c > 0$. It is conjectured that

$$\begin{aligned} f(n) &\sim c(\log n)^{-\zeta} && d = 4 \\ f(n) &\sim L(n)\, n^{-\zeta} && d < 4 \end{aligned} \tag{1.1}$$

where $\zeta = \zeta_d$ is a dimension-dependent "critical exponent" and $L = L_d$ is a slowly varying function.

It is known that

$$\begin{aligned} c_1 n^{(d-4)/2} \leqslant f(n) &\leqslant c_2 n^{(d-4)/4} && d < 4 \\ c_1(\log n)^{-1} \leqslant f(n) &\leqslant c_2(\log n)^{-1/2} && d = 4 \end{aligned} \tag{1.2}$$

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0022-4715/89/0700-0001$06.00/0 

822/56/1-2-1

(Here and throughout this paper we use $c, c_1, c_2$ for positive constants which may change from line to line.) This can be proved[7,9] by considering

$$F(n) = P\{S^1[0, n] \cap (S^2(0, n] \cup S^3(0, n]) = \emptyset\}$$

where $S^3$ is a simple random walk starting at the origin independent of $S^1$ and $S^2$ and proving

$$\begin{aligned} c_1 n^{(d-4)/2} &\leqslant F(n) \leqslant c_2 n^{(d-4)/2}, && d<4 \\ c_1(\log n)^{-1} &\leqslant F(n) \leqslant c_2(\log n)^{-1}, && d=4 \end{aligned} \tag{1.3}$$

Hence,

$$\begin{aligned} &\frac{4-d}{4} \leqslant \zeta_d \leqslant \frac{4-d}{2}, && d<4 \\ &\frac{1}{2} \leqslant \zeta_4 \leqslant 1 \end{aligned} \tag{1.4}$$

For $d=1$, one can use the results of Chapter 3 of ref. 5 to show that $f(n) \sim cn^{-1}$, i.e., $\zeta_1 = 1$, so that neither inequality in (1.4) is strict.

It has been shown[8] that $\zeta_4 = 1/2$ at least in the sense that

$$\lim_{n \to \infty} \frac{\log f(n)}{\log \log n} = -\frac{1}{2}$$

although it is still open whether $f$ has the exact asymptotic form (1.1). For $d<4$, Duplantier[2] has conjectured that the inequality (1.4) is not strict, i.e.,

$$\frac{d-4}{4} < \zeta_d < \frac{d-4}{2}, \qquad d<4$$

and has derived a nonrigorous expansion for $\zeta_d$ in $d = 4 - \varepsilon$. Duplantier and Kwon[3] have recently conjectured that $\zeta_2 = 5/8$. The conjecture comes from assuming a type of conformal invariance for the problem, concluding that only certain rational numbers are possible values for the exponent, and then using a Monte Carlo simulation to determine which value.

In this paper, we prove that the inequality (1.4) is not strict by proving that for some $\varepsilon > 0$,

$$\frac{1}{2} + \varepsilon \leqslant \zeta_2 \leqslant \frac{3}{4} \tag{1.5}$$

As can be seen by the proof, we can get an estimate on the $\varepsilon$ we derive; however, we expect it to be far from the true value. For both $d=2$ and $d=3$ we have also done Monte Carlo simulations which suggest that

$$\zeta_2 \approx 0.61..., \qquad \zeta_3 \approx 0.29$$

Our value for $\zeta_2$ is a little less than that of Duplantier and Kwon, but, as can be seen in Section 3, we can by no means rule out the possibility $\zeta_2 = 5/8$. It does seem that our simulation does not quite agree with theirs. For $d=3$, we have no proof that neither inequality in (1.4) is strict, but our value certainly supports this conjecture.

Similar results can be proved for Brownian motion and will appear in a forthcoming paper.[1] The proofs are similar; however, there does not seem to be any easy way to prove results about random walk exponents directly from Brownian motion results or vice versa.

## 2. RESULT IN TWO DIMENSIONS

In this section we prove the estimate (1.5) by proving the following.

**Theorem 1.** For $d=2$, for some $\varepsilon > 0$,

$$-\frac{3}{4} \leqslant \liminf_{n \to \infty} \frac{\log f(n)}{\log n} \leqslant \limsup_{n \to \infty} \frac{\log f(n)}{\log n} \leqslant -\frac{1}{2} - \varepsilon \tag{2.1}$$

We note that we have not proved that $f(n)$ has the form (1.1) or even that $\zeta_2$ is well defined, i.e., that

$$\lim_{n \to \infty} -\frac{\log f(n)}{\log n} = \zeta_2$$

exists.

We start with some notation: if $n$ and $r$ are nonnegative integers,

$$R_n = \{z = (z_1, z_2) \in \mathbb{Z}^2 \colon |z_i| \leqslant n\}$$

$$\partial R_n = \{z \in R_n \colon |z_1| = n \text{ or } |z_2| = n\}$$

$$R^r = R_{2^r}$$

$$\partial R^r = \partial R_{2^r}$$

and if $S$ is a simple random walk in $\mathbb{Z}^2$,

$$\xi_n = \inf\{j \geqslant 1 \colon S(j) \in \partial R_n\}, \qquad \xi^r = \xi_{2^r}$$

If $a$ is a positive real, we write $R_a$, $\partial R_a$, and $\xi_a$ for $R_{[a]}$, $\partial R_{[a]}$, and $\xi_{[a]}$, where $[\cdot]$ denotes the greatest integer function. We will prove that if

$$\tilde{J}(n) = P\{S^1[0, \xi^1_{\sqrt{n}}] \cap S^2(0, \xi^2_{\sqrt{n}}] = \varnothing\}$$

where $\xi^i_{\sqrt{n}}$ denotes the hitting time of $\partial R_{[\sqrt{n}]}$ for $S^i$, then

$$-\frac{3}{4} \leqslant \liminf_{n \to \infty} \frac{\log \tilde{J}(n)}{\log n} \leqslant \limsup_{n \to \infty} \frac{\log \tilde{J}(n)}{\log n} \leqslant -\frac{1}{2} - \varepsilon \tag{2.2}$$

A standard estimate states that for every $\alpha > 0$, there are $a = a(\alpha)$ and $c = c(\alpha) > 0$ such that

$$P\{n^{1-\alpha} \leqslant \xi_{\sqrt{n}} \leqslant n^{1+\alpha}\} = 1 - O(\exp\{-cn^a\}) \tag{2.3}$$

and hence it is immediate to conclude (2.1) from (2.2). It follows from (1.3) and (2.3) that if

$$\tilde{F}(n) = P\{S^1[0, \xi^1_{\sqrt{n}}] \cap (S^2(0, \xi^2_{\sqrt{n}}] \cup S^3(0, \xi^3_{\sqrt{n}}]) = \varnothing\}$$

then for every $\delta > 0$, there exist $c_1(\delta)$, $c_2(\delta) > 0$ with

$$c_1(\delta)\, n^{-1-\delta} \leqslant \tilde{F}(n) \leqslant c_2(\delta)\, n^{-1+\delta} \tag{2.4}$$

In fact, we can prove that

$$c_1 n^{-1} \leqslant \tilde{F}(n) \leqslant c_2 n^{-1} \tag{2.5}$$

but since (2.4) is sufficient for our proof, we will not prove (2.5). For convenience we will actually assume (2.5); the skeptical reader can easily adapt the argument so that only (2.4) is used.

*Lower Bound.* We start by proving a lemma of independent interest which relates the probability that a random walk in $\mathbb{Z}^d$ escapes a set to the (discrete) harmonic measure of the set. Let $A$ be a finite subset of $\mathbb{Z}^d$ and

$$\tau_A = \inf\{j \geqslant 1 \colon S(j) \in A\}$$

Then the harmonic measure $H_A(x)$ is defined for $x \in A$ by[11]

$$H_A(x) = \lim_{|y| \to \infty} \frac{P_y\{S(\tau_A) = x\}}{P_y\{\tau_A < \infty\}}$$

For $d = 2$, the denominator on the rhs is equal to one. For $n > 0$, let

$$\rho_{A,n} = \inf\{j \geqslant 1 \colon S(j) \in A \cup \partial R_n\}$$

**Lemma 2.** There exist constants $0 < c_1 < c_2 < \infty$ such that if $n > 0$, $A \subset R_{n/2}$, and $x \in A$, then

$$c_1 H(x)\, P_{\bar{z}}\{S(\rho) \in A\} \leqslant P_x\{S(\rho) \in \partial R_n\}\, n^{2-d}$$
$$\leqslant c_2 H(x)\, P_{\bar{z}}\{S(\rho) \in A\}$$

where $H(x) = H_A(x)$, $\rho = \rho_{A,n}$, and $\bar{z} = ([3n/4], 0, \ldots, 0)$.

*Proof.* We may assume $n \geqslant 4$, so that $\partial R_{n/2} \cap \partial R_{3n/4} = \varnothing$, $\partial R_{3n/4} \cap \partial R_n = \varnothing$. The function $\varphi(z) = P_z\{S(\rho) \in A\}$ is harmonic for $n/2 < |z| < n-1$. Hence, by the discrete Harnack inequality[10] there exist constants $0 < c_1 < c_2 < \infty$, independent of $A$ and $x$, such that for $z \in \partial R_{3n/4}$

$$c_1 P_{\bar{z}}\{S(\rho) = x\} \leqslant P_z\{S(\rho) = x\} \leqslant c_2 P_{\bar{z}}\{S(\rho) = x\} \tag{2.6}$$

By reversing paths, we can see for $x \in A$,

$$P_x\{S(\rho) \in \partial R_n\} = \sum_{y \in \partial R_n} P_y\{S(\rho) = x\} \tag{2.7}$$

Let

$$\eta = \inf\{j \geqslant 1 \colon S(j) \in \partial R_n \cup \partial R_{3n/4}\}$$

Then by the Markov property, if $y \in \partial R_n$, $x \in A$,

$$P_y\{S(\rho) = x\} = \sum_{z \in \partial R_{3n/4}} P_y\{S(\eta) = z\}\, P_z\{S(\rho) = x\} \tag{2.8}$$

Combining (2.6)–(2.8), we get

$$c_1 P_{\bar{z}}\{S(\rho) = x\} \sum_{y \in \partial R_n} P_y\{S(\eta) \in \partial R_{3n/4}\}$$
$$\leqslant P_x\{S(\rho) \in \partial R_n\}$$
$$\leqslant c_2 P_{\bar{z}}\{S(\rho) = x\} \sum_{y \in \partial R_n} P_y\{S(\eta) \in \partial R_{3n/4}\}$$

If $T(m)$ is a one-dimensional random walk and $\sigma = \inf\{j \geqslant 1 \colon T(j) = 0$ or $[n/4]\}$, then a standard estimate states that $P_0\{T(\sigma) = [n/4]\} \sim 4/n$. If we use this estimate on one component of $S$, we can derive the estimate

$$c_1 n^{d-2} \leqslant \sum_{y \in \partial R_n} P_y\{S(\eta) \in \partial R_{3n/4}\} \leqslant c_2 n^{d-2}$$

and hence

$$c_1 P_{\bar{z}}\{S(\rho) = x\}\, n^{d-2} \leqslant P_x\{S(\rho) \in \partial R_n\}$$
$$\leqslant c_2 P_{\bar{z}}\{S(\rho) = x\}\, n^{d-2} \tag{2.9}$$

A random walker starting at infinity which reaches $A$ must hit $\partial R_{3/4n}$ at some time after the last hit of $\partial R_n$ before hitting $A$. From this we can easily see that $H(x)$ is bounded above (below) by the supremum (infimum) of

$$\frac{P_z\{S(\rho)=x\}}{P_z\{S(\rho)\in A\}}$$

where the supremum (infimum) is taken over all $z\in\partial R_{3n/4}$. But by Harnack's inequality we can bound this on either side by a constant times the term with $\bar{z}$ replacing $z$, giving

$$c_1 H(x)\, P_{\bar{z}}\{S(\rho)\in A\}\leqslant P_{\bar{z}}\{S(\rho)=x\}\leqslant c_2 H(x)\, P_{\bar{z}}\{S(\rho)\in A\}$$

Substituting the above into (2.9) gives the lemma.

Although we will not need the following lemma for our main theorem, it seems appropriate to include it here.

**Lemma 3.** If $A$ is a connected subset of $\mathbb{Z}^2$ containing 0 with $\operatorname{diam}(A)\geqslant \alpha n$, then

$$P_{\bar{z}}\{S(\rho)\in A\}\geqslant c_1(1-\log\alpha)^{-1}$$

where $\bar{z}$ and $\rho$ are as in Lemma 2.

*Proof.* We may assume $\alpha\leqslant 1/2$. Let $\tilde{g}(x, y)$ be the Green's function of the random walk killed when it leaves $R_n$, i.e., if $x, y\in R_n$,

$$\tilde{g}(x, y)=\sum_{j=0}^{\infty} P_x\{S(j)=y, \xi_n>j\}$$

For $x, y\in R_{3n/4}$, a routine estimate using the local central limit theorem (see, e.g., ref. 11) gives

$$c_1\leqslant \tilde{g}(x, y)\leqslant c_2\log\frac{4n}{|x-y|} \tag{2.10}$$

Let $B\subset A$ be a subset such that $B\subset R_{\alpha n/2\sqrt{2}}$, and for each nonnegative integer $j\leqslant \alpha n/2\sqrt{2}$, $B\cap\partial R_j$ contains exactly one point. Since $A$ is a connected set containing 0 with $\operatorname{diam}(A)\geqslant \alpha n$, it is easy to see that such a $B$ exists (although $B$ might not be connected). Let

$$Y=\sum_{j=0}^{\xi_n} I\{S(j)\in B\}$$

where $I$ denotes indicator function. Then the estimate (2.10) implies that if $x \in R_{3n/4}$,

$$c_1 \alpha n \leqslant E_x(Y) \leqslant c_2 \alpha n (1 - \log \alpha)$$

(The second estimate uses the fact that there are at most $2j+1$ points in $B$ within distance $j$ of $x$.) A standard Markov argument gives

$$\begin{aligned} P_{\bar z}\{S(\rho_{A,n}) \in A\} \geqslant P_{\bar z}\{S(\rho_{B,n}) \in B\} &= \frac{E_{\bar z}(Y)}{E_{\bar z}(Y \mid Y \geqslant 1)} \\ &\geqslant \frac{E_{\bar z}(Y)}{\sup_y E_y(Y)} \\ &\geqslant c_1 (1 - \log \alpha)^{-1} \end{aligned}$$

which proves the lemma.

From Lemmas 2 and 3,

$$c_1 H(x) \leqslant P_x\{S(\rho) \in \partial R_n\} \leqslant c_2 H(x) \tag{2.11}$$

We will also need the following lemma proved by Kesten,[6] which is a discrete version of the Beurling projection theorem.

**Lemma 4.** If $A$ is a finite connected subset of $\mathbb{Z}^2$, then for every $x \in A$

$$H_A(x) \leqslant c_2 (\operatorname{diam} A)^{-1/2}$$

We are now ready to prove the lower bound of (2.2). Let $S^1, S^2, S^3$ be independent random walks in $\mathbb{Z}^2$ starting at 0 and let $\omega_1 = S^1[0, \xi^1_{\sqrt{n}}]$, $\omega_2 = S^2(0, \xi^2_{\sqrt{n}}]$, $\omega_3 = S^3(0, \xi^3_{\sqrt{n}}]$. By (2.5)

$$P\{\omega_1 \cap (\omega_2 \cup \omega_3) = \varnothing\} \geqslant c_1 n^{-1} \tag{2.12}$$

Also,

$$P\{\omega_1 \cap (\omega_2 \cup \omega_3) = \varnothing\} = P\{\omega_1 \cap \omega_2 = \varnothing\}\, P\{\omega_1 \cap \omega_3 = \varnothing \mid \omega_1 \cap \omega_2 = \varnothing\} \tag{2.13}$$

For any path $\omega_1$, Lemmas 2 and 4 give

$$P_{\omega_3}\{\omega_1 \cap \omega_3 = \varnothing\} \leqslant c_2 n^{-1/4}$$

where $P_{\omega_3}\{\omega_1 \cap \omega_3 = \varnothing\}$ denotes the probability that the walk $\omega_3$ does not intersect the fixed walk $\omega_1$. Hence

$$P\{\omega_1 \cap \omega_3 = \varnothing \mid \{\omega_1 \cap \omega_2 = \varnothing\}\} \leqslant c_2 n^{-1/4}$$

Plugging this into (2.12) and (2.13) gives

$$c_1 n^{-1} \leqslant c_2 n^{-1/4} P\{\omega_1 \cap \omega_2 = \varnothing\}$$

or

$$P\{\omega_1 \cap \omega_2 = \varnothing\} \geqslant c_1 n^{-3/4}$$

*Upper Bound.* We say a set $C \subset \mathbb{Z}^2$ disconnects $A$ and $B$ if for every $a \in A$, $b \in B$, every (nearest neighbor) path from $a$ to $b$ includes at least one point in $C \backslash (A \cup B)$.

**Lemma 5.** There exists a $c_1 > 0$ such that for every $y \in \partial R^r$, $r \geqslant 1$,

$$P_y\{S[0, \xi^{r+1}] \text{ disconnects } 0 \text{ and } \partial R^{r+1}\} \geqslant c_1$$

*Proof.* Let $S(j) = (S_1(j), S_2(j))$ be a two-dimensional simple random walk starting at 0 and let $\sigma_n = \inf\{j: S_1(j) \geqslant 3n\}$. Then it is routine to show that for some $c > 0$

$$P\left\{S_1(j) \geqslant -\frac{n}{200}, |S_2(j)| \leqslant \frac{n}{200}, 0 \leqslant j \leqslant \sigma_n\right\} \geqslant c \tag{2.14}$$

For any $x \in \mathbb{Z}^2$, let $R_n(x) = \{z + x: z \in R_n\}$.

Let $y \in \partial R^r$ and for ease suppose $y = (m, 2^r)$. Let $y_0, y_1, \ldots, y_5$ be the points $y$, $(2^r, 2^r)$, $(2^r, -2^r)$, $(-2^r, -2^r)$, $(-2^r, 2^{r-1})$, $(3 \cdot 2^{r-1}, 2^{r-1})$, respectively. Let $L_i$ be the line segment which connects $y_{i-1}$ and $y_i$ and for $a > 0$

$$B_i(a) = \{x: \operatorname{dist}(x, L_i) \leqslant a \cdot 2^r\}$$

Suppose $z_0, \ldots, z_5$ are points in $R_{n/20}(y_i)$ and $Q_1, \ldots, Q_5$ are nearest neighbor paths from $z_{i-1}$ to $z_i$ lying entirely in $B_i(1/20)$. Then it is easy to check that the path $Q = Q_1 \cdots Q_5$ never hits 0 or $\partial R^{r+1}$ and makes a "loop" about 0 disconnecting 0 and $\partial R^{r+1}$.

Let

$$\tau_i = \inf\{j: S(j) \in R_{in/100}(y_i)\}$$

Then by (2.14), for any $z \in R_{(i-1)n/100}(y_{i-1})$,

$$P_z\{S(j) \in B_i(1/20), 0 \leqslant j \leqslant \tau_i\} \geqslant c$$

and hence the probability that the simple random walk starting at $y$ makes a path $Q_1 \cdots Q_5$ as above is at least $c^5$, which completes the proof.

We note that the above construction allows us to get estimates on the probability $c_1$. However, since we do not expect these estimates to be close to the actual value, we will not do it here.

**Lemma 6.** There exist $\alpha > 0$, $c_2 < \infty$ such that

$$P_0\{S(0, \xi_n) \text{ disconnects } \{0\} \text{ and } \partial R_n\} \geqslant 1 - c_2 n^{-\alpha}$$

*Proof.* It suffices to prove the lemma for $n = 2^r$. By Lemma 5, for each $s < r$

$$P_0\{S(\xi^s, \xi^{s+1}) \text{ disconnects } \{0\} \text{ and } \partial R^r \,|\, S[0, \xi^s]\} \geqslant c_1$$

and hence

$$\begin{aligned} P_0\{S(0, \xi^r) \text{ does not disconnect } \{0\} \text{ and } \partial R^r\} &\leqslant (1 - c_1)^{r-1} \\ &= c_2 n^{-\alpha} \end{aligned}$$

where $c_2 = (1 - c_1)^{-1}$ and $\alpha = -\log(1 - c_1)/\log 2$.

We are now ready to prove the upper bound. As before, let $\omega_1 = S^1[0, \xi^1_{\sqrt{n}}]$, $\omega_2 = S^2(0, \xi^2_{\sqrt{n}}]$, $\omega_3 = S^3(0, \xi^3_{\sqrt{n}}]$. Let $B = \{\omega_1 \colon \omega_1$ does not disconnect 0 and $\partial R_{\sqrt{n}}\}$. By Lemma 6, $P(B) \leqslant c_2 n^{-\alpha/2}$. It is also clear that $P_{\omega_2}\{\omega_1 \cap \omega_2 = \varnothing \,|\, \omega_1 \in B^c\} = 0$. Consider the random variable

$$X(\omega_1) = P_{\omega_2}\{\omega_1 \cap \omega_2 = \varnothing\}$$

Then, since $\omega_2$ and $\omega_3$ are independent,

$$[X(\omega_1)]^2 = P_{\omega_2, \omega_3}\{\omega_1 \cap (\omega_2 \cup \omega_3) = \varnothing\}$$

and hence by (2.5) and Schwarz inequality,

$$\begin{aligned} c_2 n^{-1} &\geqslant P\{\omega_1 \cap (\omega_2 \cup \omega_3) = \varnothing\} \\ &= E_{\omega_1}(X^2) \\ &\geqslant [E_{\omega_1}(X I_B)]^2 \, [E_{\omega_1}(I_B)]^{-1} \\ &= [E_{\omega_1}(X)]^2 \, [P(B)]^{-1} \end{aligned}$$

which gives

$$P\{\omega_1 \cap \omega_2 = \varnothing\} = E_{\omega_1}(X) \leqslant c_1 n^{-1/2 - \alpha/4}$$

## 3. SIMULATIONS IN TWO AND THREE DIMENSIONS

In order to estimate $\zeta_d$ for $d=2, 3$, Monte Carlo simulations were made of the probability

$$h(n)=P\{S^1(i)\neq S^2(j)\colon 0\leqslant i\leqslant n,\ 0\leqslant j\leqslant n,\ (i,j)\neq(0,0)\}$$

While this is not exactly the same as $f(n)$, we expect that $h(n)$ has asymptotic form

$$h(n)\sim L(n)\,n^{-\zeta} \tag{3.1}$$

where $L$ is a slowly varying function which should be asymptotic to a constant times the $L$ in (1.1) and $\zeta$ is the same as in (1.1). Suppose that $M$ independent pairs of random walks are taken, and let $K(n)$ be the number of such pairs which have no intersection up through time $n$. Then we can estimate $h(n)$ by $M^{-1}K(n)$. To estimate an exponent such as $\zeta$, we must assume that $h$ has a form such as (3.1) and that the asymptotic regime has been reached. Let us suppose for the moment that

$$h(n)=c_1 n^{-\zeta} \tag{3.2}$$

Then if $n_1<n_2$,

$$\zeta=\frac{\log p}{\log n_1-\log n_2}$$

where $p=p(n_1,n_2)=h(n_2)/h(n_1)$. Since the walks are independent, an approximate 95% confidence interval for $p$ would be $[\bar{p}_-,\bar{p}_+]$, where $\bar{p}=K(n_2)/K(n_1)$ and

$$\bar{p}_\pm=\bar{p}\pm 2\left[\frac{\bar{p}(1-\bar{p})}{K(n_1)}\right]^{1/2}$$

and hence an approximate 95% confidence interval for $\zeta$ would be $[\zeta(\bar{p}_+),\zeta(\bar{p}_-)]$, where

$$\zeta(\bar{p}_\pm)=\frac{\log\bar{p}_\pm}{\log n_1-\log n_2}$$

Also, if $n_1<n_2<n_3$, the estimate for $\bar{p}(n_1,n_2)$ is essentially independent of the estimate for $\bar{p}(n_2,n_3)$. Of course, this assumes that $h(n)$ has the form (3.2), but if $h(n)$ has the asymptotic form (3.1), this should not be too far from the correct estimate.

Table I

| $n_1$ | $n_2$ | $K(n_1)$ | $\zeta(\bar{\rho}_+)$ | $\zeta(\bar{\rho})$ | $\zeta(\bar{\rho}_-)$ |
|---|---|---|---|---|---|
| $d=2$ | | | | | |
| 50 | 70 | 182,727 | 0.603 | 0.610 | 0.618 |
| 70 | 100 | 148,814 | 0.605 | 0.613 | 0.621 |
| 100 | 150 | 119,595 | 0.604 | 0.613 | 0.621 |
| 150 | 200 | 93,296 | 0.602 | 0.612 | 0.623 |
| 200 | 250 | 78,224 | 0.604 | 0.617 | 0.631 |
| 250 | 300 | 68,158 | 0.608 | 0.623 | 0.639 |
| 300 | 350 | 60,837 | 0.617 | 0.635 | 0.652 |
| 350 | 400 | 55,167 | 0.594 | 0.613 | 0.632 |
| 400 | 450 | 50,832 | 0.570 | 0.591 | 0.612 |
| 450 | 499 | 47,414 | 0.592 | 0.615 | 0.639 |
| $d=3$ | | | | | |
| 50 | 100 | 224,147 | 0.288 | 0.291 | 0.294 |
| 100 | 200 | 183,175 | 0.286 | 0.289 | 0.293 |
| 200 | 300 | 149,912 | 0.280 | 0.284 | 0.289 |
| 300 | 400 | 133,582 | 0.283 | 0.289 | 0.295 |
| 400 | 500 | 122,926 | 0.282 | 0.289 | 0.296 |
| 500 | 600 | 115,252 | 0.277 | 0.285 | 0.293 |
| 600 | 700 | 109,414 | 0.282 | 0.290 | 0.299 |
| 700 | 800 | 104,626 | 0.275 | 0.285 | 0.294 |
| 800 | 900 | 100,725 | 0.275 | 0.285 | 0.295 |
| 900 | 999 | 97,404 | 0.275 | 0.285 | 0.296 |

For $d=2$, $M=3{,}000{,}000$ pairs of random walks of length 500 were generated; for $d=3$, $M=1{,}000{,}000$ pairs of walks of length 1,000 were generated. We list the results for various values of $n_1$, $n_2$ in Table I.

From the table we can see that the value of $\zeta_3$ seems to be about 0.29, and that we are in the asymptotic regime. There is considerably more variance for the values of $\zeta_2$, which indicates that either the asymptotic regime has not been reached or perhaps that the asymptotic behavior of $h(n)$ is more complicated than (3.1). We find it curious that our interval for $n_1=50$, $n_2=70$ does not include 5/8. The values for $n_1=50$, $n_2=70$ were considered by Duplantier and Kwon[3] in deriving the $\zeta_2=5/8$ conjecture, and their simulations place the exponent in a range $0.622 \pm 0.004$. Our simulations would tend to indicate that $\zeta_2 < 5/8$, but we certainly cannot preclude $\zeta_2 = 5/8$ with any degree of confidence.

## ACKNOWLEDGMENTS

The work of K.B. was supported by NSF grant DMS-8702620. The work of G.F.L. was supported by NSF grant DMS-8702879 and an Alfred P. Sloan Research Fellowship.


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